

CQC

Composing Quantum Channels

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Quantum channel:

$$\rho \rightarrow T \rightarrow \rho' = T(\rho)$$

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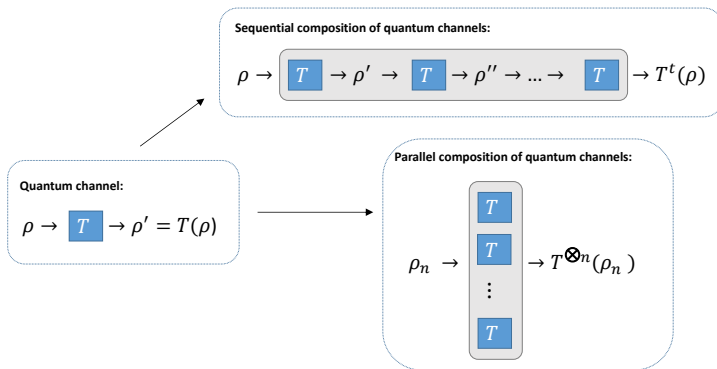
Sequential composition of quantum channels:

$$\rho \rightarrow \boxed{T} \rightarrow \rho' \rightarrow \boxed{T} \rightarrow \rho'' \rightarrow \dots \rightarrow \boxed{T} \rightarrow T^t(\rho)$$

Quantum channel:

$$\rho \rightarrow \boxed{T} \rightarrow \rho' = T(\rho)$$

Motivation



O1: Theory and tools

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O2: Implications

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O2: Implications

O3: Dissemination

WP1: Single
Channels

O1: Theory and tools

O2: Implications

O3: Dissemination

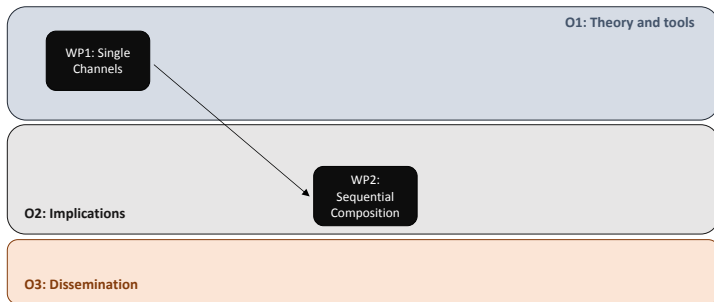
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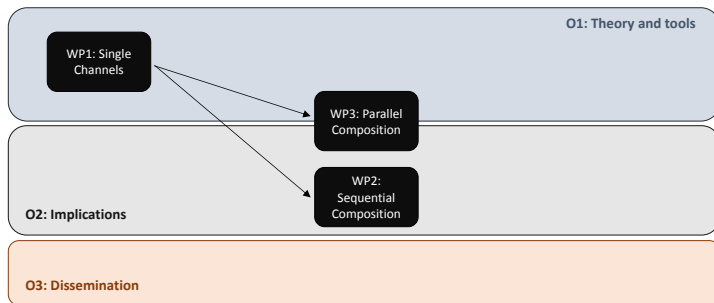
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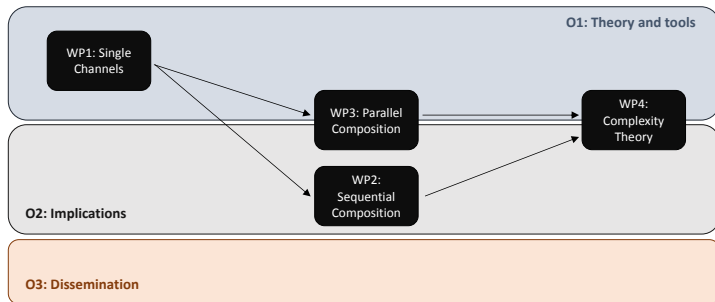
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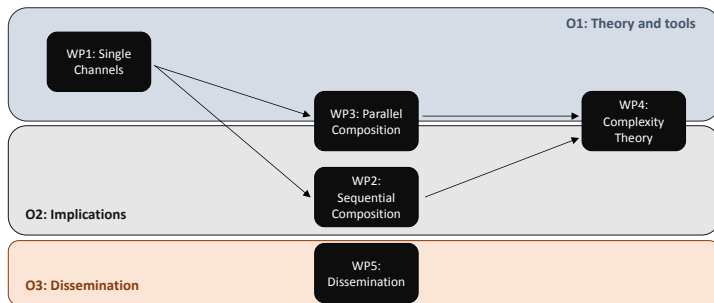
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Prof. M. Wolf

Motohisa Fukuda
Daniel Reitzner
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Volkher Scholz





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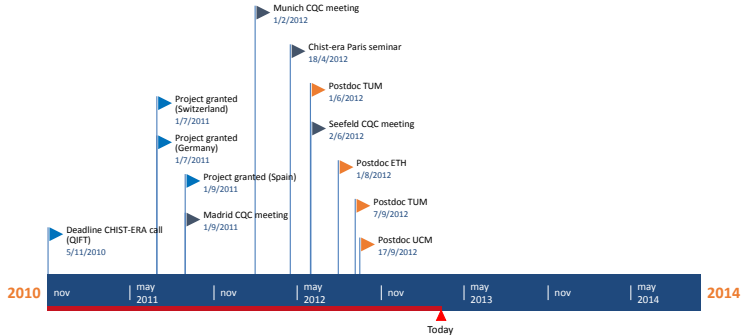
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Timeline



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(un)decidable problems in Quantum Information

$$\left[\begin{array}{c} \phi \\ \rho \\ \{T_i\}_{i=1}^k \end{array} \right]$$

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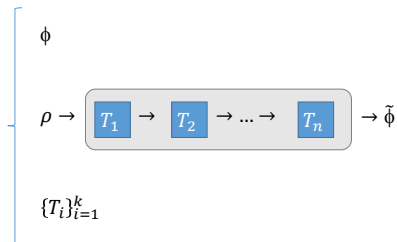
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MM Wolf, TS Cubitt, D Perez-Garcia

<http://arxiv.org/abs/1111.5225>

(un)decidable problems in Quantum Information



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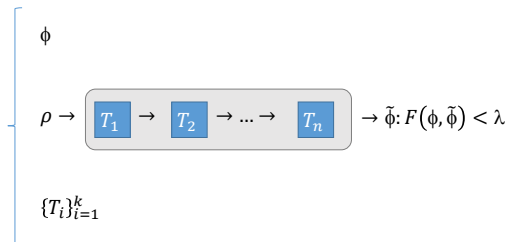
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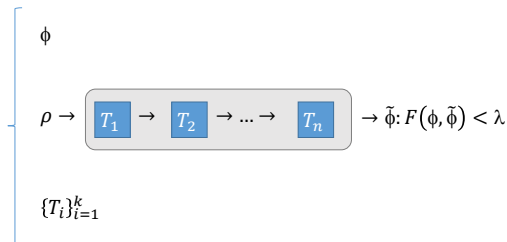
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(un)decidable problems in Quantum Information



Theorem

Let (k, d) be $(2, 5)$, $(5, 3)$ or pointwise larger, take $0 < \lambda < 1$ and $|\phi\rangle \in \mathbb{C}^d$. Then, there is no algorithm which upon input $\rho \in \mathcal{M}_d(\mathbb{C})$ and of a set of quantum channels $\{T_i : \mathcal{M}_d(\mathbb{C}) \rightarrow \mathcal{M}_d(\mathbb{C})\}_{i=1}^k$ decides if $\exists n \geq 1$ and a sequence $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ for which

$$\langle \phi | T_{i_1} \cdots T_{i_n}(\rho) | \phi \rangle < \lambda.$$

Principle of Maximum Entropy in Spin Chains

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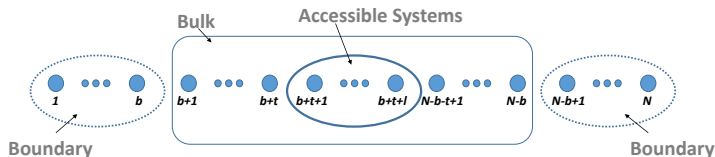
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Comm. Math. Phys. (2013)

B Collins, C Gonzalez-Guillen, D Perez-Garcia

<http://arxiv.org/abs/1201.6324>

Entanglement and Long-range Interactions

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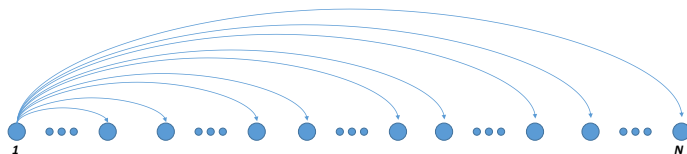
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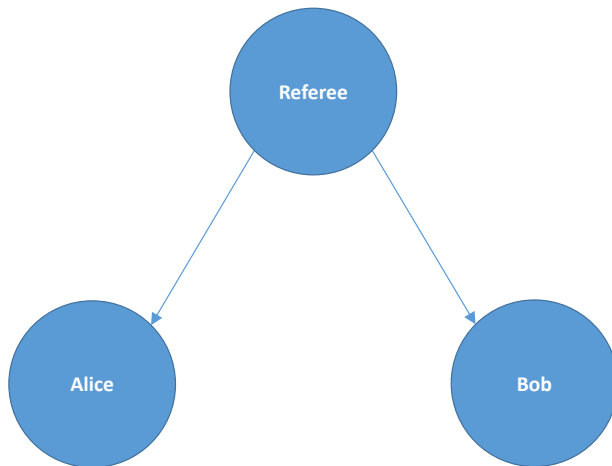
Phys. Rev. B 87, 035114 (2013)

A Cadarso, D Perez-Garcia, MM Wolf, JI Cirac, D Perez-Garcia

<http://arxiv.org/abs/1209.3998>

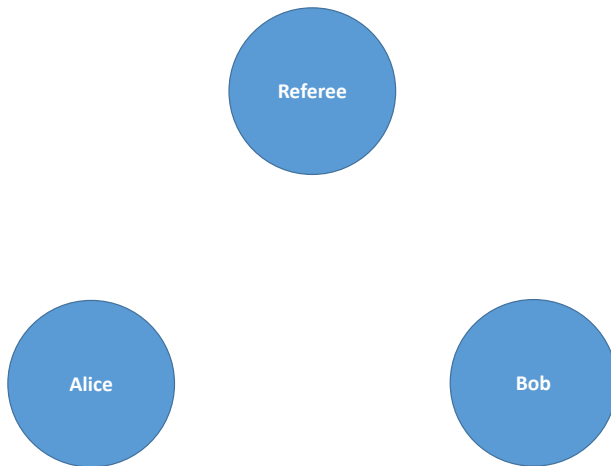
Rank-one Quantum Games

1. Charlie prepares $|\psi\rangle^{ABR} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$ and sends the registers A and B to Alice and Bob.



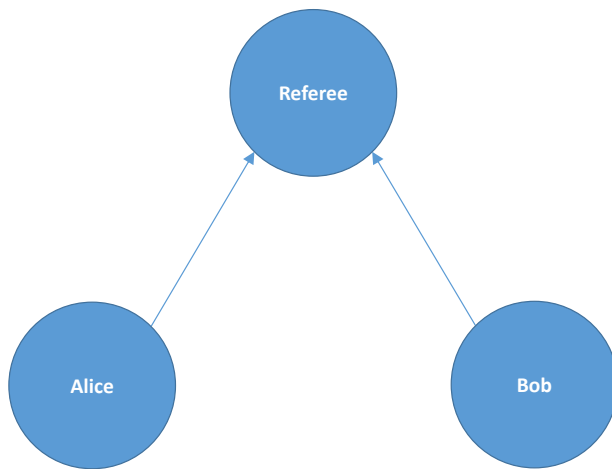
Rank-one Quantum Games

2. Alice and Bob share $|\chi\rangle^{A'B'} \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$. They apply local unitaries $U_{AA'}$ and $V_{BB'}$: $(U_{AA'} \otimes V_{BB'} \otimes \text{id}_R) |\psi\rangle |\chi\rangle$.



Rank-one Quantum Games

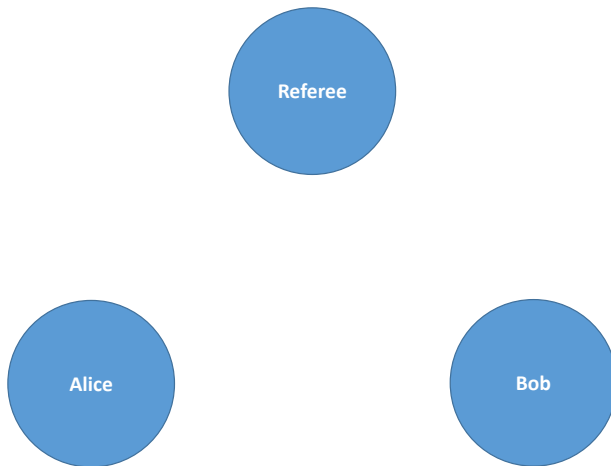
3. Alice and Bob send back registers A and B .



Rank-one Quantum Games

4. Charlie applies the projective measurements:

$$\{P_{\text{accept}} = |\gamma\rangle\langle\gamma|, P_{\text{reject}} I - P_{\text{accept}}\}.$$



Definition

The supremum over all states $|\chi\rangle$ and quantum operations $U_{AA'}$ and $V_{BB'}$ of the probability that Alice and Bob win the game is called the value of the game. This value is denoted by $\omega^*(G)$.

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- $\omega^*(G)$ can be approximated up to a constant factor using SDPs.

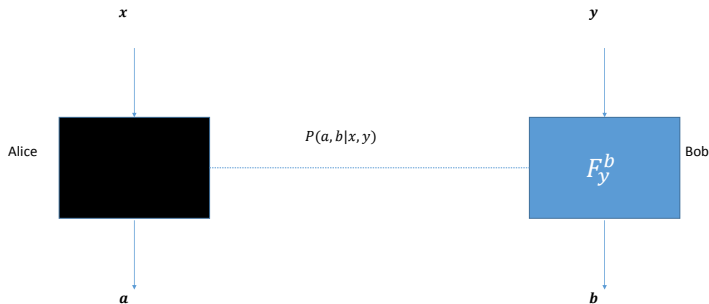
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- ▶ $\omega^*(G)$ can be approximated up to a constant factor using SDPs.
- ▶ $\omega^*(G)$ does not verify perfect parallel repetition. In fact, the violation can be unbounded.

$$\omega^*(G_n^{\otimes 2}) \geq \frac{n}{2} \omega^*(G_n)^2$$

Quantum Steering and Space-like Separation



Phys. Rev. Lett. (2013)
M Navascues, D Perez-Garcia
<http://arxiv.org/abs/1204.6220>

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Quantum Shannon Theory

Basic question of (Quantum) Shannon Theory:

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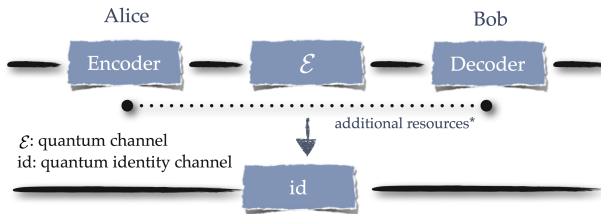
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Basic question of (Quantum) Shannon Theory:

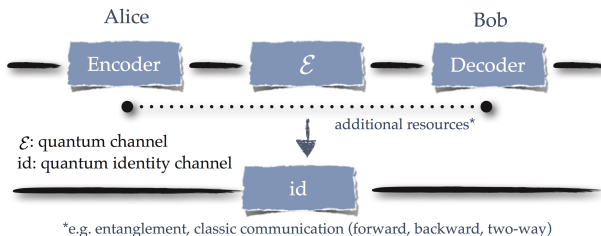
How many uses m of some channel $\mathcal{E}$ are necessary in order to simulate n copies of the identity channel?



*e.g. entanglement, classic communication (forward, backward, two-way)

Basic question of (Quantum) Shannon Theory:

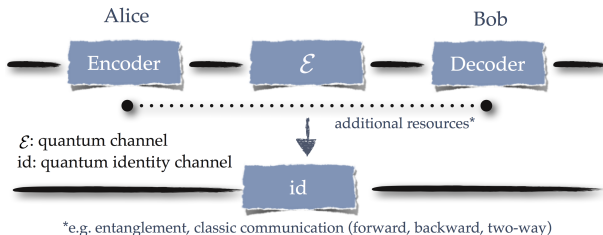
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Answer: Quantum Capacity $\mathcal{Q}(\mathcal{E})$

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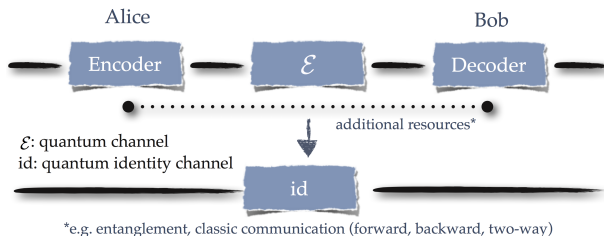
Answer: Quantum Capacity $\mathcal{Q}(\mathcal{E})$

Known formula: $\mathcal{Q}(\mathcal{E}) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{Q}(\mathcal{E}^{\otimes n})$ for

$$\mathcal{Q}(\mathcal{E}) = \max_{\phi} [S(B)_{(\text{id} \otimes \mathcal{E})(\phi)} - S(AB)_{(\text{id} \otimes \mathcal{E})(\phi)}]$$

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$\rightsquigarrow$ Expression via information theoretic quantities.

Channel Simulations

Reverse question:

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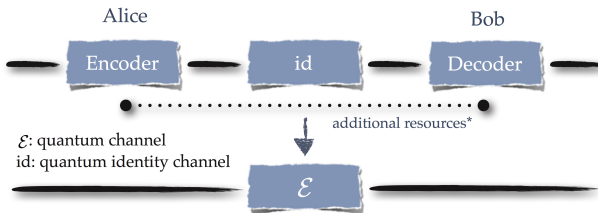
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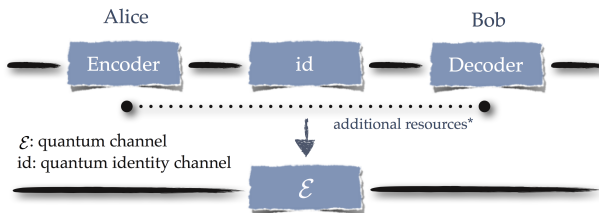
How many uses n of the identity channel are necessary to simulate m copies of some quantum channel $\mathcal{E}$?



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How many uses n of the identity channel are necessary to simulate m copies of some quantum channel $\mathcal{E}$?



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If entanglement is for free, the rate equals the **entanglement assisted quantum capacity**. If classical communication is free, the rate is called **Entanglement Cost** $E_C(\mathcal{E})$. If $\mathcal{E}$ is a quantum to classical channel, and shared randomness is given, the rate is called **Information Gain** $C_{IG}(\mathcal{E})$.

C. Bennett, P. Shor, J. Smolin, A. Thapliyal, IEEE T. Inf. Th. 48 (2002); C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, A. Winter, arxiv:0912.5537 (2009); M. Berta, M. Christandl, R. Renner, Comm. Math. Phys. 306 (2011)

Channel Simulations

Using classical communication, at what rate can the quantum identity channel simulate a quantum channel?

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Using classical communication, at what rate can the quantum identity channel simulate a quantum channel?

Theorem (Entanglement cost of quantum channels)

$E_C(\mathcal{E}) = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\psi^n} E_F((\mathcal{E}^{\otimes n} \otimes id)(\psi^n))$ where E_F denotes the **entanglement of formation** $E_F(\rho_{AB}) = \inf_{\{p_i, \rho_i\}} \sum_i p_i S(A)_{\rho_i}$

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At what rate can the classical identity channel simulate a quantum measurement (using shared randomness)?

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At what rate can the classical identity channel simulate a quantum measurement (using shared randomness)?

Theorem (Universal measurement compression)

$C_{IG}(\mathcal{E}) = \max_{\rho} I(X : R)_{M \otimes \mathcal{I}(\rho)}$ and measurement simulation is possible if $c \geq C_{IG}(\mathcal{E})$, $c + r \geq \max_{\rho} H(X)_{M(\rho)}$, c : classical communication, r : shared randomness.

Using classical communication, at what rate can the quantum identity channel simulate a quantum channel?

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Remark: This characterizes the amount of information gained by a measurement, which is interesting for physical applications.

M. Berta, M. Christandl, F. Brandao, S. Wehner, Inf. Th. Proc. (ISIT) (2012); ²M. Berta, J. M. Renes, M. M. Wilde, arxiv:1301.1594 (2013); ³A. Winter, Comm. Math. Phys. 244 (2004)

Result 1: Convergence of Quantum Markov Chains

Sequential composition of quantum (or classical) channels:

$$\rho \xrightarrow{\mathbf{T}} \rho' \xrightarrow{\mathbf{T}} \rho'' \xrightarrow{\mathbf{T}} \dots \xrightarrow{\mathbf{T}} \rho^{(n)} = \mathbf{T}^{(n)}(\rho)$$

Result 1: Convergence of Quantum Markov Chains

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Theorem (Jordan-Bound)

For some $C > 0$, independent of n , and d_μ size of the largest Jordan block for μ we have $\|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq C \mu^{n-d_\mu+1} n^{d_\mu-1}$

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Theorem (Schur-Bound)

$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\|_\diamond \leq 2d^{3/2}(\mu + 2d^{1/2})^{d^2-1}n^{d^2-1}\mu^{n-d^2+1}.$$

Result 1: New Spectral Convergence Bound

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Result 1: New Spectral Convergence Bound

Theorem (Szehr, Reeb, Wolf '13)

Suppose $\|\mathcal{T}^n\| \leq C \forall n \in \mathbb{N}$. Let $m = m_{\mathcal{T}-\mathcal{T}_\infty}$ be minimal polynomial and μ spectral radius of $\mathcal{T} - \mathcal{T}_\infty$. Then, for $n > \frac{\mu}{1-\mu}$ we have

$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq \mu^n R(\mu, m, n) \prod_{m/(z-\lambda_D)} \frac{1 - (1 + \frac{1}{n})\mu|\lambda_i|}{\mu - |\lambda_i| + \frac{\mu}{n}},$$

where $R(\mu, m, n) = \frac{4Ce^2\sqrt{|m|}(|m|+1)}{(1-(1+\frac{1}{n})\mu)^{3/2}}.$

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- Exponential improvement compared to Schur-bound.

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$$\|\mathcal{T}^n - \mathcal{T}_\infty^n\| \leq \mu^n R(\mu, m, n) \prod_{m/(z-\lambda_D)} \frac{1 - (1 + \frac{1}{n})\mu|\lambda_i|}{\mu - |\lambda_i| + \frac{\mu}{n}},$$

where $R(\mu, m, n) = \frac{4Ce^2\sqrt{|m|(|m|+1)}}{(1-(1+\frac{1}{n})\mu)^{3/2}}$.

- ▶ Exponential improvement compared to Schur-bound.
- ▶ Contains the Jordan-Bound as a Corollary, but with specified constants.

Result 2: Tensor-stable positivity

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- ▶ But it is **positive** of course: $\vartheta_n(\rho) \geq 0$ for $\rho \geq 0$.
- ▶ Furthermore it is **tensor-stable positive**, i.e. $\vartheta_n^{\otimes k} = \vartheta_{nk}$ is a positive map for all $k \in \mathbb{N}$.

Result 2: New Capacity-Bounds from Tensor-Stable positivity

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Result 2: New Capacity-Bounds from Tensor-Stable positivity

Theorem (Müller-Hermes, Reeb, Wolf '12)

Let $\mathcal{M} : \mathfrak{M}_n \rightarrow \mathfrak{M}_n$ be an **invertible, unital, tensor-stable positive map**, which is **not completely positive**. Then

$$\mathcal{Q}(\mathcal{T}) \leq \frac{\log(\|\mathcal{T} \circ \mathcal{M}^{*-1}\|_{\diamond} \|\mathcal{M}^*(I)\|_{\infty}) \log(n)}{\log(\|\vartheta \circ \mathcal{M} \circ \vartheta\|_{\diamond})}$$

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- ▶ We do not know yet, whether there are non-trivial tensor-stable positive maps.
- ▶ Progress in this direction would also shed new light on the NPT-bound entanglement problem.

Result 3: Are some problems of QIT undecidable?

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- ▶ Important consequence: **If the problem is undecidable, there cannot exist a computable formula, that answers general cases.**

Result 3: Fidelity Reachability Problem

Standard Question in Control Theory: Given controls $\mathcal{T}_1, \dots, \mathcal{T}_k : \mathcal{M}_d \rightarrow \mathcal{M}_d$, can one reach a certain state ρ_f from a certain input state ρ_i .

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Corollary (Buchinger, Wolf '13)

This is undecidable if (k, d) is $(3, 6), (6, 4), (8, 3)$ or pointwise larger.

Thank you for your attention.

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